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SELF-SIMILAR SOLUTIONS OF ONE-DIMENSIONAL PROBLEMS OF GAS FILTRATION WITH A QUADRATIC RESISTANCE LAW

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One-dimensional problems of isothermal gas filtration in a porous medium with a quadratic resistance law are considered. For the cases of plane and axial symmetry of the problem under constant initial conditions, analytical expressions for the gas pressure and velocity are obtained and an empirical formula for calculating the gas pressure is proposed.

Problems of gas filtration through a cracked and porous medium arise in investigating a number of technological processes: separation of mixtures by the methods of gas chromatography, gas fracture of seams, propagation of gaseous products in a rock mass in underground detonation of explosives, etc. To describe the motion of a gas in a cracked-porous medium, use is made of the system of equations of continuum mechanics: the continuity equation, the equation of motion, and the energy equation. However, empirical laws that are established experimentally and relate the pressure gradient to the velocity of motion – Darcy's law or Forchheimer's law (binomial law of filtration) – are used as the equation of motion instead of the Euler equation. The resistance force that is produced in the motion of a gas in a porous medium is determined by the properties of the medium and the filtered flow. The dependence of the resistance force on the gas velocity can be considered linear for small Reynolds numbers (Re < 1) and quadratic for large numbers (Re > 10).

Below we consider the case of a quadratic resistance law. For plane and cylindrical symmetries of the problem, we are able to find here analytical expressions for the gas pressure and velocity. In the case of plane symmetry, an empirical formula to calculate the gas pressure (with a background pressure that is more than 10% of the pressure of the inflowing gas) is proposed.

System of Filtration Equations and Permissible Group of Transformations. Gas filtration in a porous medium with large local Reynolds numbers is described by a continuity equation, a quadratic dependence of the pressure gradient on the velocity, and an equation of state:

$$\frac{\partial \rho}{\partial t} + \text{div}\ (\rho \vec{u}) = 0\,, \quad \nabla P = -\frac{1}{b}\rho\ |\vec{u}|\ \vec{u}\,, \quad P = c^2 \rho\,. \tag{1}$$

In the case of one-dimensional (plane, axisymmetric, spherically symmetric) flow of the gas, system (1) is written in the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{x^{\nu}} \frac{\partial}{\partial x} (\rho u x^{\nu}) = 0\,, \quad \frac{\partial P}{\partial x} = -\frac{\rho}{b} u^2\,, \quad P = c^2 \rho\,. \tag{2}$$

The system of equations of one-dimensional filtration (2) was investigated numerically in [1].

Eliminating the gas density from (2), we obtain the equation

$$u u_t - K u_x + \frac{3}{2} u^2 u_x - K \frac{\partial}{\partial x}\left(\frac{u}{x}\right) = 0\,, \quad K = \frac{bc^2}{2}\,. \tag{3}$$

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Let us find the group of transformations of the form

$$x' = f_1(x, t, u, a), \quad t' = f_2(x, t, u, a), \quad u' = f_3(x, t, u, a) \tag{4}$$

such that upon substitution of the variables $(x, t, u) \to (x', t', u')$ Eq. (3) remains invariant [2]. According to a standard scheme, for this purpose it is necessary to find expressions for the coordinates of the tangential vector field corresponding to the variables x, t, and u that are determined as

$$\xi^1 = \left.\frac{\partial f_1}{\partial a}\right|_{a=0}, \quad \xi^2 = \left.\frac{\partial f_2}{\partial a}\right|_{a=0}, \quad \eta = \left.\frac{\partial f_3}{\partial a}\right|_{a=0}.$$

The coordinates of the tangential vector field that correspond to a change in the derivatives are expressed in terms of ξ^1, ξ^2, and η by the formulas

$$\xi_l = D_l \eta - u_k D_l \xi^k \quad (l = 1, 2 \text{ for } \frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial t} \text{ respectively}),$$

$$\xi_{11} = D_1 \xi_1 - u_{xx} D_1 \xi^1 - u_{xt} D_1 \xi^2 \quad (\text{for } \frac{\partial^2 u}{\partial x^2}).$$

Here

$$D_1 = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u}, \quad D_2 = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u}.$$

If we take

$$E \equiv u u_t - K u_{xx} + \frac{3}{2} u^2 u_x - K \frac{\partial}{\partial x}\left(\frac{u}{x}\right) = 0,$$

then from the determining equation

$$\underset{1}{X}E\,|_{E=0} = \xi^1 \frac{\partial E}{\partial x} + \xi^2 \frac{\partial E}{\partial t} + \eta \frac{\partial E}{\partial u} + \xi_1 \frac{\partial E}{\partial u_x} + \xi_2 \frac{\partial E}{\partial u_t} + \xi_{11} \frac{\partial E}{\partial u_{xx}} = 0$$

we can find ξ^1, ξ^2, and η. As a result we obtain the following formulas for the coordinates of the tangential vector field:

$$\xi^1 = 2C_1 x^1 + C_2 \delta_{\nu,0}, \quad \xi^2 = 3C_1 x^2 + C_3, \quad \eta^1 = -C_1 u^1. \tag{5}$$

Thus, the transformation group permissible by Eq. (3) is independent of the parameter K and of ν when $\nu \neq 0$. In the case $\nu = 0$, solution of (3) can be sought in the coordinates of a traveling wave.

Knowledge of the coordinates of the tangential vector field enables us to obtain an explicit form of transformations (4). Let us consider the quantities that remain constant upon substitution of the variables (5), i.e., the invariants of the group of transformations (4) for Eq. (3). Relation (3) can be written in terms of these invariants and as a result can be reduced to an ordinary differential equation.

When $\nu = 0$ the basis of the Lie algebra is formed by the operators

$$X_1 = \frac{\partial}{\partial x}; \quad X_2 = \frac{\partial}{\partial t}; \quad X_3 = u\frac{\partial}{\partial u} - 2x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t}.$$

For the linear combination $X = \alpha X_2 + X_1$, the independent invariants are $I_1 = u$, $I_2 = x - \alpha t$, i.e., Eq. (3) has a solution in the coordinates of a traveling wave: $I_1 = f(I_2)$. Hence it follows that $u = f(\theta)$, where $\theta = x - \alpha t$, α is some characteristic velocity.

Self-Similar Solutions of the System of Filtration Equations for $\nu = 0, 1$. We find the invariants of Eq. (3) relative to the group of transformations of the tension X_3:

$$X_3 = u\frac{\partial}{\partial u} - 2x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t}.$$

Using the formula

$$\frac{du}{u} = -\frac{dx}{2x} = -\frac{dt}{3t},$$

we obtain two independent first integrals. As the basis we select the variables $I_1 = xt^{-2/3}$ and $I_2 = ut^{1/3}$. Then a solution of Eq. (3) can be written in the form $u = t^{-1/3}\varphi(xt^{-2/3})$. If we employ the boundary conditions

$$P(x = 0, t) = P_1, \quad \lim_{r\to 0}(Pux) = At^{1/3} \tag{6}$$

for the plane and axisymmetric cases, respectively, the pressure of a gas in a porous medium will also depend only on the self-similar variable $xt^{-2/3}$. Going over to the renormalized self-similar variable

$$\theta = x\left(\frac{4}{9bc^2t^2}\right)^{1/3}, \tag{7}$$

we have the following formulation of the problem:

$$\frac{1}{\theta^\nu}\frac{d}{d\theta}(f\varphi\theta)^\nu = \frac{1}{\theta}\frac{df}{d\theta}, \quad \frac{df}{d\theta} = -f\varphi^2,$$

$$f(\theta\to\infty) = N = P_0/P_1, \tag{8}$$

$$P(x = 0, t) = P_1\ (\nu = 0), \quad \lim_{x\to 0}(Pux) = At^{1/3}\ (\nu = 1).$$

Here f and φ are dimensionless analogs of the gas pressure and velocity that are related to P and u by the following dependences:

$$P = P_1^* f(\theta); \quad u = \sqrt[3]{2bc^2/3t}\,\varphi(\theta);$$

$$P_1^* = P_1\ (\nu = 0); \quad P_1^* = A\sqrt[3]{2/3b^2c^4}\quad (\nu = 1). \tag{9}$$

From system of equations (8) we find the equation for φ

$$\frac{d\varphi}{d\theta} - \varphi^3 + \theta\varphi^2 + \frac{\nu\varphi}{\theta} = 0. \tag{10}$$

When $\nu = 0$, we make in (10) the substitution

$$\varphi = (0.5\theta^2 - 2^{1/3}z)^{-1},$$

and then, considering θ to be a dependent variable, from (10) we obtain

$$\frac{d\theta}{dz} = 2^{-1/3}\theta^2 - 2^{2/3}z = \frac{2^{1/3}}{\varphi}. \tag{11}$$

If we take

$$\theta = -2^{2/3} y^{-1} \frac{dy}{dz}, \tag{12}$$

then equality (11) becomes the Airy equation for y:

$$y'' - zy = 0, \tag{13}$$

whose general solution is the combination of the Airy functions [3]

$$y = B_0 A_i(z) + A_0 B_i(z). \tag{14}$$

When $\nu = 1$, we make the substitution

$$\varphi = (\theta^2 - 2^{2/3}\theta\psi)^{-1}, \quad \theta = 2^{1/3}(\psi^2 - z)^{-1},$$
$$\frac{d\psi}{dz} = \frac{2^{1/3}}{\theta} = \psi^3 - z, \tag{15}$$

from which, for $\psi = -y^{-1}dy/dz$, we also arrive at Eq. (12).

Knowing the velocity of the gas φ, we can easily find the expression for its pressure

$$f = \frac{y^2}{\varphi\theta^{2\nu}}, \quad \nu = 0, 1. \tag{16}$$

From formula (16) it can be seen that $y^2(z)$ coincides with the expression for a gas flux in a porous medium when $\nu = 0$.

In the case $\nu = 0$, from (11) and (16) we have

$$f(z) = 2^{1/3}\left[\left(\frac{dy}{dz}\right)^2 - zy^2\right], \tag{17}$$

and, using the initial and boundary conditions of system (8), we find that the solution $y(z)$ is determined on the interval $[z_0, z_1]$, where z_0 and z_1 correspond to $\theta = 0$ and $\theta \to \infty$. From (12) it follows that

$$y'(z_0) = 0, \quad y(z_1) = 0, \tag{18}$$

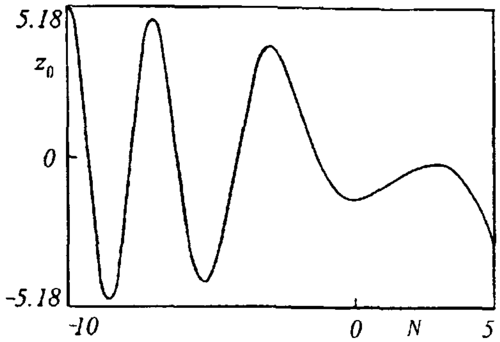
while from (11) and (17) we can obtain the equalities

$$y(z_0) = \sqrt{\left(-\frac{1}{z_0}\right)}; \quad y'(z_1) = \sqrt{\left(\frac{N}{2}\right)}. \tag{19}$$

This and expression (14) yield the system of equations

$$B_0 A_i'(z_0) + D_0 B_i'(z_0) = 0, \quad B_0 A_i(z_0) + D_0 B_i(z_0) = \sqrt{\left(-\frac{1}{z_0}\right)},$$
$$B_0 A_i(z_1) + D_0 B_i(z_1) = 0, \quad B_0 A_i(z_1) + D_0 B_i(z_1) = \sqrt{\left(\frac{N}{2}\right)}. \tag{20}$$

The Wronskian of the Airy functions A_i and B_i is equal to



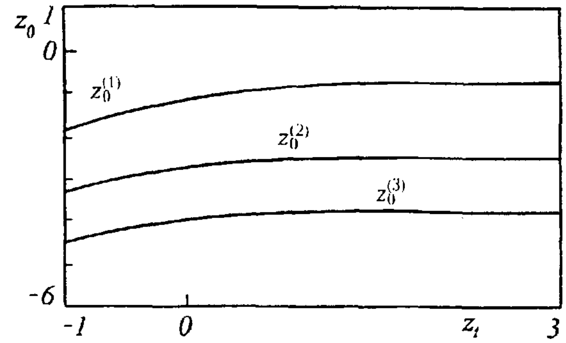


Fig. 1. Function that determines the dependence of z_0 on N for z_1 = 3.0. All the quantities are dimensionless.

Fig. 2. Dependence $z_0(z_1)$ for different selected roots z_0 ($\nu = 0$). z_0 and z_1 are dimensionless.

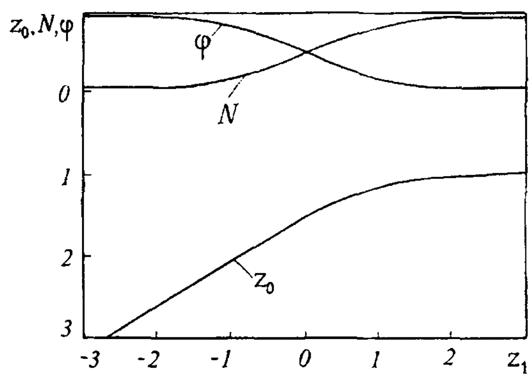


Fig. 3. Maximum root z_0, N, and $\varphi(\theta = 0)$ vs. z_1 for plane motion of a gas in a porous medium ($\nu = 0$). All the quantities are dimensionless.

$$w\,[A_i(z)\,,\,B_i(z)] = \frac{1}{\pi}\,.$$

Knowing it, we can express the constants A_0 and D_0

$$A_0 = -\pi \cdot 2^{-1/6}\sqrt{N}\,B_i(z_1)\,,\quad D_0 = \pi \cdot 2^{-1/6}\sqrt{N}\,A_i(z_1)\,. \tag{21}$$

from (20). Then (20) will become the system

$$R(z_0, z_1) \equiv B_i(z_1)\,A_i'(z_0) - A_i(z_1)\,B_i'(z_0) = 0\,,$$

$$B_i(z_0)\,A_i(z_1) - A_i(z_0)\,B_i(z_1) = \sqrt{\left(\frac{-2^{1/3}}{\pi^2 N z_0}\right)}\,. \tag{22}$$

From these equations with a the prescribed N we can obtain z_0 and z_1. If (22) is considered to be the dependences $z_0(z_1)$ and $N(z_0(z_1), z_1)$ different z_1 will correspond to different boundary conditions of problem (8).

The dependence $z_0(z_1)$ determined by equalities (22) turns out to be nonunique. Figure 1 presents the function whose roots correspond to z_0 for z_1 = 3.0. The figure shows that z_0 can be determined variously. The characteristic form of functions $z_0(z_1)$ with different selected roots is shown in Fig. 2. (In what follows we use a root of 1.)

Figure 3 demonstrates the dependences of the maximum root $z_0(z_1)$, the relative pressure at infinity $N(z_1)$, and the gas velocity $\varphi(\theta = 0)$ in the case of plane motion of the gas in a porous medium. For each $N \in [0, 1]$, we can find the corresponding z_1, and therefore the system of equations (8) for $\nu = 0$ can be solved for all N.

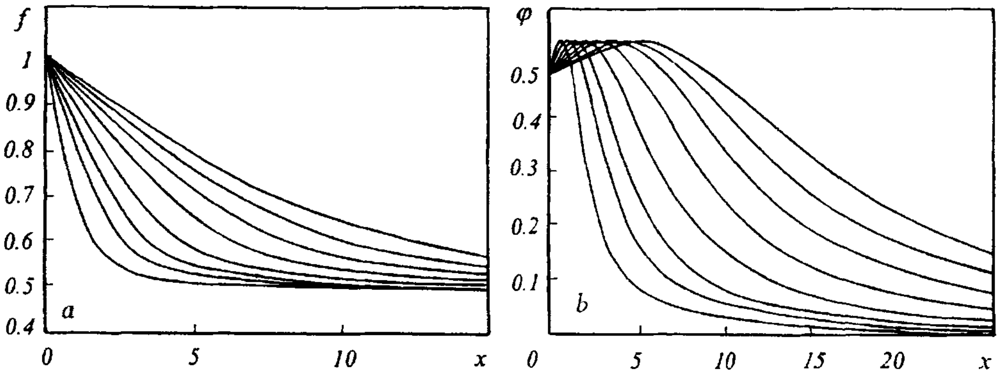


Fig. 4. Dimensionless pressure of a gas f (a) and dimensionless velocity of the gas φ (b) vs. the space variable x with a relative background pressure $N = 0.5005$ for $t = 0$, 2, 3, 5, 8, 12, 16, and 21 sec (the curves are ordered from the bottom upward) ($\nu = 0$). f and φ are dimensionless; x, m.

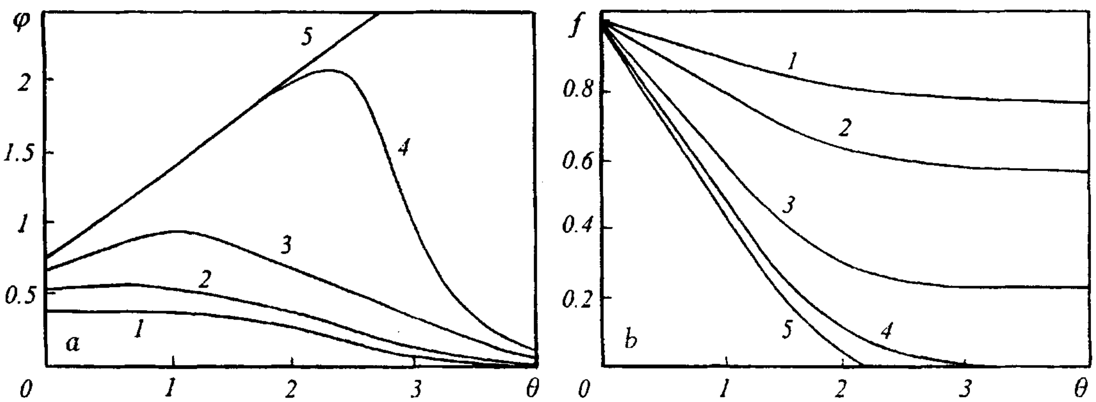


Fig. 5. Velocity φ (a) and dimensionless pressure f (b) vs. the self-similar variable ($\nu = 0$). φ, f, and θ are dimensionless.

Using (14), (16), and (21) and the expression for φ when $\nu = 0$, we can easily determine that the pressure of the gas in a porous medium, its velocity, and the self-similar variable are expressed in terms of the variable z by the formulas

$$f(z) = 2^{1/3}(y'(z)^2 - zy^2(z)), \quad \varphi(z) = 2^{-1/6}y^2(z) f^{-1}(z),$$

$$\theta(z) = -2^{-1/3}\frac{B_i(z_1)A_i'(z) - A_i(z_1)B_i'(z)}{B_i(z_1)A_i(z) - A_i(z_1)B_i(z)},$$

$$y(z) = 2^{-1/6}\pi N^{1/2}(B_i(z_1)A_i(z) - A_i(z_1)B_i(z)).$$

Figure 4 shows plots of the gas pressure and velocity versus the space variable at different instants for a relative background pressure $N = 0.5005$ (we took $4/(9bc^2) = 1$).

Figure 5a illustrates the velocity φ as a function of the self-similar variable at $N = 0$, $5 \cdot 10^{-4}$, 0.22, 0.55, 0.75 (curves 1-5, respectively). The pressure of the gas for the same N in plane motion is shown in Fig. 5b. In Fig. 5a, it can be seen that there are qualitative differences in the behavior of the dependences of the velocity of gas motion in a porous medium for $N = 0$ and $N > 0$. In the case of gas filtration into a vacuum, the velocity of motion increases with distance, $\varphi \to \infty$ when $\theta \to \infty$ according to the law $\varphi \sim \theta + \theta^{-2}$. For a nonzero initial pressure of the gas, there is a maximum in the velocity of motion.

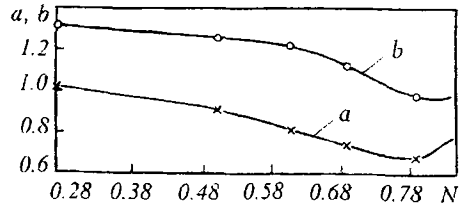


Fig. 6. Behavior of the parameters of the empirical formula a and b as a function of N ($\nu = 0$). a, b, and N are dimensionless.

We can show that the extremum point of the velocity of gas motion $\varphi(\theta)$ corresponds to the surface between the gas inflowing into the porous medium and the displaced gas. The law of conservation of mass for the displaced gas that initially filled the porous medium is written in the form

$$N\theta_0 = \int_{\theta_0}^{\infty} (f - N)\, d\theta .$$

Here θ_0 is the self-similar coordinate of the surface between the gases. Integrating this equation by parts and using (8), we obtain

$$f(\theta_0)\,\theta_0 = -\int_{0}^{\infty} f'\theta d\theta = f(\theta_0)\,\varphi(\theta_0),$$

from which $\varphi(\theta_0) = \theta_0$, and therefore (10) yields $d\varphi/d\theta = 0$ when $\theta = \theta_0$.

In the case of plane symmetry of problem (8) when the relative pressure at infinity N is not very low (for $N < 0.1$), the relationship between the gas pressure and the time and space variables can be represented in the form of an exponential dependence on the self-similar variable θ, i.e., there is the formula

$$f(\theta) \cong N + (1 - N)\exp[-a(N)\,\theta^{b(N)}],\quad \theta = x\left(\frac{4}{9bc^2t^2}\right)^{1/3}. \tag{23}$$

For example, with $N = 0.506$ $a = 0.809$ and $b = 1.213$ (see Fig. 6).

In the case $\nu = 1$, the equations that determine z_0 and z_1 have the form

$$y(z_0) = 0,\quad \psi^2(z_1) = z_1 . \tag{24}$$

Using the initial and boundary conditions of problem (8), we obtain a system of equations to determine the constants z_0, z_1, A_0, and D_0:

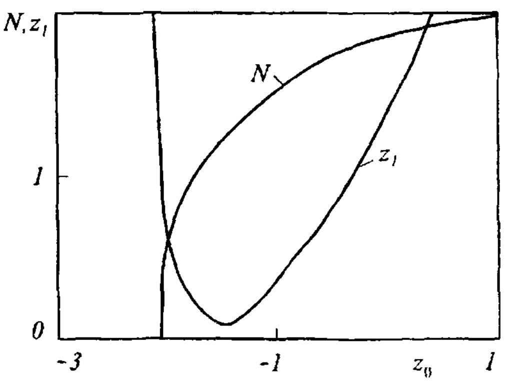
$$\begin{gathered}[B_i(z_0)\,A_i'(z_1) - A_i(z_0)\,B_i'(z_1)]^2 = z_1\,[B_i(z_0)\,A_i(z_1) - A_i(z_0)\,B_i(z_1)]^2,\\ [B_i(z_0)\,A_i'(z_1) - A_i(z_0)\,B_i'(z_1)]^2 = z_1 N 2^{-1/3}\pi^{-2},\\ A_0 = \pi\cdot 2^{-1/6}\,B_i(z_0),\quad D_0 = -\pi\cdot 2^{-1/6}\,A_i(z_0).\end{gathered} \tag{25}$$

From the first equation of system (25) we can determine the dependence $z_1(z_0)$. Using it, from the second equation of (25) we can find z_0 and the constants A_0 and D_0 for each N.

Figure 7 shows the dependences $N(z_0)$ and $z_1(z_0)$. For each $N > 0$, the corresponding value of z_0 can be found, and therefore problem (8) is also solved for $\nu = 1$.

The expressions for the pressure and velocity of the gas are written in the form

$$f(z) = y^2(z) + 2^{1/3}y'(z)\,([y'(z)]^2 - zy^2(z))/y(z),\quad \varphi(z) = y^2(z)\,\theta^{-2}(z)\,f^{-1}(z),$$



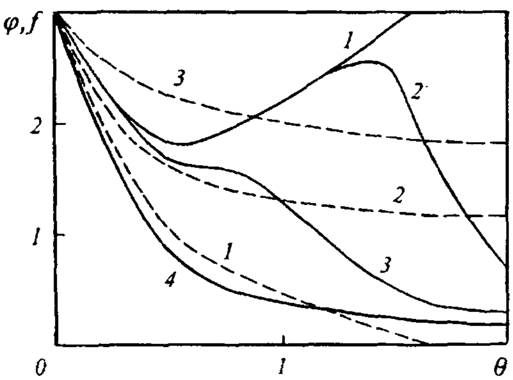


Fig. 7. Dependences $N(z_0)$ and $z_1(z_0)$ for $\nu = 1$. N, z_1, and z_0 are dimensionless.

Fig. 8. Dependence $\varphi(\theta)$ (the solid lines) for $N = 0$, 10^{-5}, 0.05, 1.0 (curves 1-4) and dependence $f(\theta)$ (the dashed lines) for $N = 0$, 1.0, 1.62 (curves 1-3) in the case $\nu = 1$. φ, f, and θ are dimensionless.

$$\theta(z) = 2^{-1/3}\left([y'(z)]^2 - zy^2(z)\right)/y^2(z),$$
$$y(z) = 2^{-1/6}\pi\,[A_i(z)\,B_i(z_0) - A_i(z_0)\,B_i(z)]. \tag{26}$$

Figure 8 presents the dependences $\varphi(\theta)$ and $f(\theta)$ for different N.

When $z_1 \to \infty$ the solution in the limit describes the motion of a gas in a porous medium with zero background pressure $N = 0$ [4]. As in the plane case, there are qualitative differences in the behavior of the velocity for $N = 0$ and $N > 0$. In the first case, the velocity of motion increases with distance, $\varphi \to \infty$ when $\theta \to \infty$ according to the law $\varphi \sim \theta + 2\theta^{-2}$, and in the second case it decreases to zero.

Analysis of Eq. (10) shows that, for $N = 0$, the velocity φ has a minimum, for $0 < N < N_0$, where $N_0 \approx 0.05$, it has a minimum and a maximum, and, for $N > N_0$, the velocity φ decreases monotonically to zero as θ increases.

As in the case $\nu = 0$, for axisymmetric gas motion, the boundary between the inflowing gas and the displaced gas corresponds to the point θ_0, where $\varphi(\theta_0) = \theta_0$.

By means of the maximum of the filtration rate we can evaluate the time of gas motion through a fixed layer of the porous medium.

The obtained analytical and empirical dependences of the dynamic characteristics of a gas filtered in a porous medium at a high rate can be used to check numerical solutions in mathematical modeling of gas motion with allowance for a quadratic resistance law and to calculate directly problems of gas filtration.

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NOTATION

Dimensional quantities: A, constant used in the boundary condition in the case of axial symmetry of the problem, $\text{kg}\cdot\text{m}\cdot\text{sec}^{-8/3}$; c, isothermal velocity of sound, m/sec; K, parameter in the equation for the gas velocity, m^2/sec^2; P, pressure of the gas, Pa; P_0, background pressure, Pa; P_1, boundary pressure for $x = 0$ in the case of plane symmetry of the problem, Pa; t, time variable, sec; $\vec{u}$ and u, vector and scalar velocities of gas motion, m/sec; x, space variable, m; ρ, gas density, kg/m^3. Dimensionless quantities: a, parameter of the one-parameter transformation group; $a(N)$ and $b(N)$, coefficients of the approximation expression for the dimensionless pressure of the gas; $A_i(z)$ and $B_i(z)$, Airy functions; B_0 and D_0, constants; b, coefficient of resistance in filtration; D_1 and D_2, total-differentiation operators; N, pressure of the gas at infinity; X, X_i, infinitesimal operators; y, dependent variable; z, independent variable; z_0 and z, specific values of the variable z that correspond to $\theta = 0$ and $\theta \to \infty$; $\delta_{\nu,0}$, Kronecker symbol; η, coordinate of the tangential vector field that corresponds to the gas velocity; ν, index of symmetry of the problem ($\nu = 0$, 1, and 2 for plane, axisymmetric, and spherically symmetric motion of the gas, respectively); ξ^1, coordinate of the tangential vector field that corresponds to the space variable; ξ^2, coordinate of the tangential vector field that corresponds to the time variable; θ, self-similar variable; θ_0, self-similar coordinate of the surface between the gases; ψ, dependent variable.

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